Exercise 39

In Exercise 9.1.15 we formulated a model for learning in the form of the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where $P(t)$ measures the performance of someone learning a skill after time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression?

Solution

This differential equation is separable, so we can solve for $P(t)$ by bringing all terms with P to the left and all constants and terms with t to the right and then integrating both sides.

$$\begin{aligned} dP &= k(M - P)\,dt \\ \frac{dP}{M - P} &= k\,dt \\ \int \frac{dP}{M - P} &= \int k\,dt \end{aligned}$$

Use a u-substitution to solve the integral on the left.

$$\begin{aligned} \text{Let } u &= M - P \\ du &= -dP \quad \to \quad -du = dP \end{aligned}$$

$$\begin{aligned} \int \frac{-du}{u} &= \int k\,dt \\ \ln|u| &= -kt + C \\ e^{\ln|M-P|} &= e^{-kt+C} \\ |M - P| &= e^{-kt}e^{C} \\ M - P &= \pm e^{C} e^{-kt} \end{aligned}$$

Let $C_1 = \pm e^C$. Then

$$P(t) = M - C_1 e^{-kt}.$$

To find the limit of this expression we take the limit of $P(t)$ as t goes to infinity.

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \left(M - C_1 e^{-kt}\right) \\ &= \lim_{t\to\infty} M - \lim_{t\to\infty} C_1 e^{-kt} \\ &= M - C_1 \underbrace{\lim_{t\to\infty} e^{-kt}}_{=\,0} \\ &= M - 0 \end{aligned}$$

Thus,

$$\lim_{t\to\infty} P(t) = M.$$

This makes sense. After a really long time a person will reach his or her maximum level of performance.